mum and minimum values reached by θ on each cycle, θ_{max} and θ_{min}, one obtains

$$\cos^2\theta_{max} = \frac{C}{(B - C)h^2}(2TB - h^2) \qquad (7)$$

$$\cos^2\theta_{min} = \frac{C}{(A - C)h^2}(2TA - h^2) \qquad (8)$$

After differentiating, the following equations are obtained:

$$2\sin\theta_{max}\cos\theta_{max}\dot{\theta}_{max} = -\frac{2\dot{T}CB}{(B - C)h^2} \qquad (9)$$

$$2\sin\theta_{min}\cos\theta_{min}\dot{\theta}_{min} = -\frac{2\dot{T}CA}{(A - C)h^2} \qquad (10)$$

Comparison of Eq. (9) and (10) with (4) indicates that the form is identical with Eq. (4), if B and A, respectively, are substituted for I, and C for J. The similarity of form when θ_{max} or θ_{min} is used as a variable indicates that the techniques used for the symmetric case (e.g., in Refs. 1, 2, 4, and 5) can be applied with little difficulty to the asymmetric case. Of course, the asymmetry will still affect the calculation of $\dot{T}$.

With $\dot{T}$ negative, the above equations indicate that both $\dot{\theta}_{max}$ and $\dot{\theta}_{min}$ are positive. Thus for $A > B > C$ (i.e., minimum moment of inertia about the spin axis) the spin axis deviation from an inertial reference increases with time and the body attitude is said to be unstable.

For the case $C > B > A$, $h^2 > 2TB$, which corresponds to a body spinning about its axis of largest inertia with a superimposed wobble, the equations for the rate of change of θ_{max} and θ_{min} can be shown to be identical to those for $A > B > C$ [Eqs. (9) and (10)]. With $C > B > A$ and $\dot{T}$ negative, the attitude angular rates $\dot{\theta}_{max}$ and $\dot{\theta}_{min}$ become negative, indicating a stable attitude condition where the spin axis approaches an inertially fixed direction.

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Flow Field of an Exhaust Plume Impinging on a Simulated Lunar Surface

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ONE of the problems associated with space exploration is determining the effects caused by the impingement of an exhaust plume on a foreign surface. Roberts, in a recent paper,[1] investigated the problem of a single axisymmetric jet impinging on a dust-covered lunar surface. He presented an approximate method for calculating the exhaust-plume flow field. However, this method is accurate only when the square of the nozzle exit Mach number is much greater than one ($M_j^2 \gg 1$) and the descent height is large. This note presents a less restricted method for calculating the exhaust-plume flow field and compares results with experimental data.

Figure 1 shows a sketch of a lunar vehicle with its retrorocket impinging on a lunar surface. The exhaust gases of interest expand after leaving the nozzle and pass through a shock wave just above the lunar surface. Analytically the problem can be divided into the supersonic region ahead of the shock and the subsonic and supersonic regions behind it. The flow properties ahead of the shock can be calculated using the method of characteristics. The region behind it could probably best be calculated using an inverse method, similar to the one used to calculate the flow region around the nose of a blunt body. However, this would be difficult due to the high nonuniformity of the upstream flow. For now a simpler approach is preferable.

Newtonian theory as usually applied assumes a uniform upstream flow. However, for nonuniform flow it is still valid if applied locally. For this problem the upstream properties vary considerably with location, and, therefore, the local flow properties ahead of the shock must be determined. It is preferable to know the shock location although approximate results can be obtained by assuming that it lies on the lunar surface. Reference 1 presents a method for calculating the shock shape once the upstream flow field is known.

The flow properties on the lunar surface can then be calculated as follows:

1) The exhaust-plume flow field is calculated using the method of characteristics and assuming that the lunar surface is not present.

2) The shock location is calculated using the method presented in Ref. 1, or it is assumed to lie on the lunar surface.

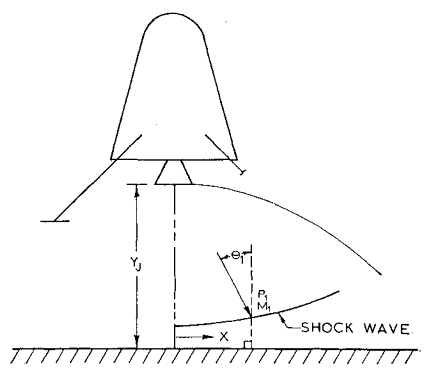
3) The flow properties just ahead of the shock are obtained using results from the method of characteristics.

4) The lunar-surface pressures are calculated using Newtonian theory,

$$p_s/P_c = (P_{T2}/p_1)(p_1/P_c)\cos^2\theta_1$$

where

p_1 = pressure just ahead of shock
p_s = lunar-surface static pressure
P_c = rocket-chamber total pressure
θ_1 = angle between the flow direction and the normal to the lunar surface
$\frac{P_{T2}}{p_1}$ = ratio of the total pressure behind a normal shock, at the point of interest, to the static pressure ahead of it



d_T = THROAT DIAMETER OF NOZZLE
Y_J = DISTANCE FROM NOZZLE EXIT
X = DISTANCE FROM NOZZLE CENTERLINE

Fig. 1 Lunar model

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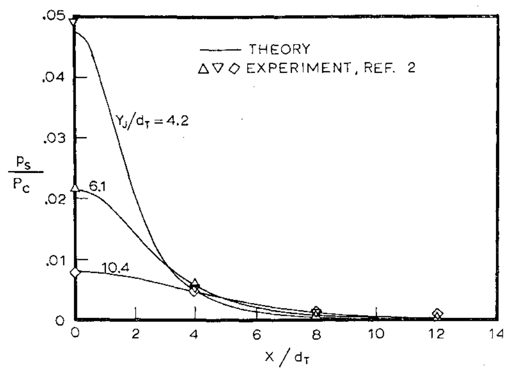

Fig. 2 Pressure distributions on the lunar surface for flow from the sonic nozzle

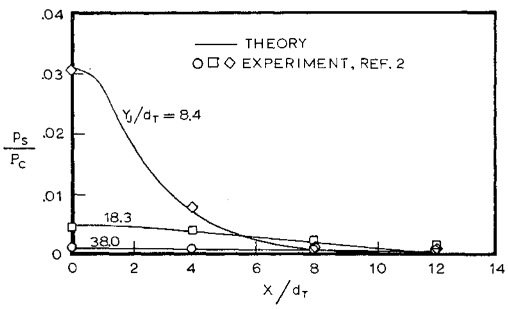

Fig. 3 Pressure distributions on the lunar surface for flow from the conical nozzle

For an ideal gas

$$\frac{P_{T2}}{p_1} = \left[\frac{(\gamma + 1)M_1^2}{2}\right]^{[\gamma/(\gamma-1)]} \left[\frac{\gamma + 1}{2\gamma M_1^2 + 1 - \gamma}\right]^{[1/(\gamma-1)]}$$

5) The other surface properties are calculated by noting that the surface entropy is constant and equal to the lunar-surface stagnation point value.

Reference 2 presents results from an experimental study to determine the effects of an exhaust plume on a simulated lunar surface. Lunar-surface pressure distributions for flow from cold air jets were measured at various descent heights. Several nozzles were used; however, only results for a single sonic nozzle and a single 4:1, 15° conical nozzle ($M_j = 2.94$) are used here. Figures 2 and 3 show experimental surface pressure distributions obtained using these two nozzles.

The theoretical values shown in these figures were calculated using the method presented here. A recently completed method of characteristics program[3] was used to calculate the jet exhaust plume. In order to obtain answers as accurately as possible, the shock shapes were obtained from schlieren photographs taken during the experiments of Ref. 2.

Referring to Figs. 2 and 3, it can be seen that there is excellent agreement between theory and experiment. If the shock shape had been assumed to lie on the lunar surface, the theoretical pressures would have been lower. This assumption was made in Ref. 2 and explains why the theoretical lunar pressures at the centerline of the jet were lower than the experimental values.

It is of interest to compare results with those calculated using the method of Ref. 1. Table 1 shows a comparison of centerline pressures for the conical nozzle. As can be seen, the method of Ref. 1 accurately predicts the centerline pressure at the larger values of descent height. At the lowest value the pressure is underpredicted; however, that method is expected to become less accurate as the descent height decreases.

Table 1 Comparison of centerline pressures for flow from the conical nozzle

	$(p_s/P_c)_{x/d_T=0}$		
Y_J/d_T	Theory this note	Theory Ref. 1	Experiment Ref. 2
8.4	0.0305	0.0189	0.0300
18.3	0.0047	0.0040	0.0045
38.0	0.0009	0.0009	0.0009

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Techniques for the Derivation of Element Stiffness Matrices

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REFERENCE 1 has given a matrix formulation of the "unit load theorem" approach to the derivation of structural element stiffness properties. The present note shows that there are two alternative approaches to such determinations and presents a matrix formulation of both. One approach is based on a direct formulation of the desired relationships without recourse to work or strain energy principles, whereas the other is an application of the principle of virtual displacements. A brief comparison of all three alternatives also will be given. A detailed derivation and critical review of the three approaches appears in Ref. 2.

The notation of Ref. 1 will be retained to the extent possible. Consider first the "direct formulation" technique, the general concepts of which apply to derivations based on assumed stress or strain distributions or assumed displacements. To retain correspondence with Ref. 1, which restricts its attention to assumed stress distributions, one begins by considering a stress vector $\{\bar{\sigma}\}$ ($= \{\bar{\sigma}_{xx} \ldots \bar{\tau}_{zx}\}$), the terms of which are approximated by functions whose coefficients are the constants $\{k_1\}$. These assumed functional relationships for the stresses can be written in matrix form as

$$\{\bar{\sigma}\} = [\bar{U}]\{k_1\} \tag{1}$$

where the terms of $[\bar{U}]$ are the variables in these relationships and are generally dimensional variables.

By use of the appropriate stress-strain relationships and subsequent integration of the strain-displacement equations, relationships for the displacements Δ are obtained in terms of the coefficients (k_1,k_2), where the added coefficients k_2 pertain to rigid body motion terms in the displacement functions. Evaluation of these displacement relationships at the points where stress resultants will be assumed to act provides the following set of algebraic equations (with $\{k\} = \{k_1 : k_2\}$):

$$\{\Delta\} = [B]\{k\} \tag{2}$$

hence

$$\{k\} = [B]^{-1}\{\Delta\} \tag{3}$$

Since $[B]$ must be a square matrix, it is apparent that the present development is limited to cases where the coefficients (k_1) of the assumed stress distribution, plus the coefficients

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